Of Shapes and Numbers

A love story between Topology and Algebra

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What is Algebra?

- Oxford Dictionary:
  "**algebra**,noun:
  the part of mathematics in which letters and other general symbols are used to represent numbers and quantities in formulae and equations"
- Wikipedia:
  "In algebra, which is a broad division of mathematics, **abstract algebra** (occasionally called modern algebra) is the study of algebraic structures. "
- What are **algebraic structures** then?

What are algebraic structures?

Definition

An **algebraic structure** consists of a set A and a collection of **operations** $(\mu_1, ..., \mu_n)$ of arities $(a_1, ..., a_n) \in \mathbb{N}^n$:

$$\mu_k : A^{a_k} \to A,$$

where A^{a_k} denotes the a_k-fold catesian product of A with itself and $A^0 = *$ is the singleton. Further we have a set of **axioms**, that these operations need to satisfy.

Examples of Algebraic structures

- **Groups**:
  G is a set, $(\cdot, 1, ^{-1})$ are operations with arities $(2, 0, 1)$ meaning:

$$\begin{array}{ccc} \cdot : G \times G \to G, & 1 : \{*\} \to G, & ^{-1} : G \to G \\ (g, h) \mapsto g \cdot h & 1 \in G & g \mapsto g^{-1} \end{array}$$

  satisfying **associativity, unitality** and **inversibility**

$$\begin{array}{ccc} (g \cdot h) \cdot k = g \cdot (h \cdot k) & 1 \cdot g = g & g \cdot g^{-1} = 1 \\ & g \cdot 1 = g & g^{-1} \cdot g = 1 \end{array}$$

- **Monoids** M with operations $(\cdot, 1)$ and arities $(2, 0)$ satisfying associativity, unitality.

Examples of Algebraic structures

- **rings** R with operations $(+, 0, -, \cdot, 1)$ and arities $(2, 0, 1, 2, 0)$ satisfying group axioms $(+)$, monoid axioms $(\cdot)$ and **distributivity**.
- **fields** F with operations $(+, 0, -, \cdot, 1, ^{-1})$ and arities $(2, 0, 1, 2, 0, 1)$ satisfying two sets of group axioms and **distributivity**.
- **lattices** L with operations $(\cup, \cap)$ with arities $(2, 2)$ satisfying **absorption law**.
- **bounded lattices** L with operations $(\cup, \top, \cap, \bot)$ with arities $(2, 0, 2, 0)$ a lattice with **maximum and minimum** .
- **Boolean algebras**
- **Vector spaces**
- **Algebras** A a vectorspace with a multiplication.
- **associative Algebras**
- ...

Equations and their solutions

From its first moments a big part of algebra was dedicated to find solutions to (polynomial) equations.

- **Diophantine equations**: Working over rings or particularly the integers $\mathbb{Z}$
- Formulae for solutions of polynomials in degree 2,3,4
- There are no Formulae for solutions of general polynomials in degree ≥ 5! (Galois theory)
- **Algebraic Geometry**: Exploring the geometry of these solutions sets

Theorem (The Fundamental Theorem of Algebra)

Any polynomial $P \in \mathbb{C}[x]$ with coefficients in the field of complex numbers $\mathbb{C}$ of degree $n \geq 1$ has a root in $\mathbb{C}$.

What is Topology?

- Oxford Dictionary:
  "**topology**,noun:
  the way the parts of something are arranged and related"
- Wikipedia:
  "In mathematics, **topology** (from the Greek words $\tau o \pi o \sigma$, 'place, location', and $\lambda o \gamma o \sigma$, 'study') is concerned with the properties of a geometric object that are preserved under continuous deformations, such as stretching, twisting, crumpling, and bending; that is, without closing holes, opening holes, tearing, gluing, or passing through itself.
  "
- Mathematically: "Topology is the study of **topological spaces** and continuous functions."

What are topological spaces?

Definition

A **topological space** is a pair (X, τ), where X is a set and τ collection of subsets of X satisfying axioms:

1. The empty set and X itself belong to τ: $\emptyset \in \tau, X \in \tau$.
2. Any finite intersection of members of τ is in τ:

$$A_1, ..., A_n \in \tau \Rightarrow \bigcap_{i=1}^{n} A_i \in \tau$$

3. Any arbitrary union of members of τ is in τ:

$$A_i \in \tau, i \in I \Rightarrow \bigcup_{i \in I} A_i \in \tau$$

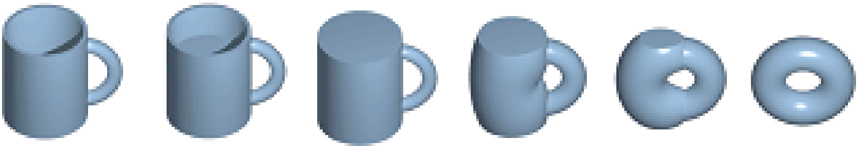
What are continuous maps?

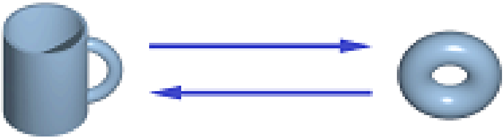
Definition

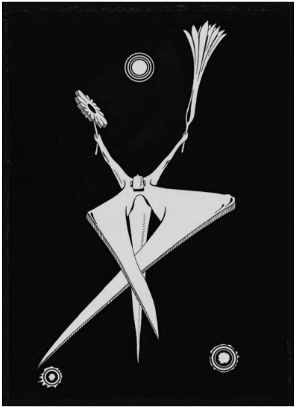
Give two topological spaces $(X, \tau), (Y, \sigma)$ a **continuous map** is a map $f : X \to Y$ such that $f^{-1}(A) \in \tau, \forall A \in \sigma$.
If f is bijective and f as well as f^{-1} continuous, we call it a **homeomorphism**. In that case we call X and Y **homeomorph** and write $X \cong Y$

Homeomorphy is the notion of equality in topology, as isomorphy is in algebra.
Two homeomorphic spaces are generally speaking "the same" space!

Dougnuts and Coffee mugs







Strengths and Weaknesses of Topology

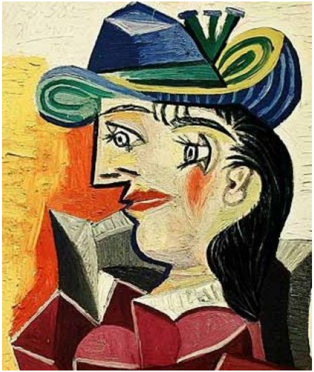
Topology is bad to describe:

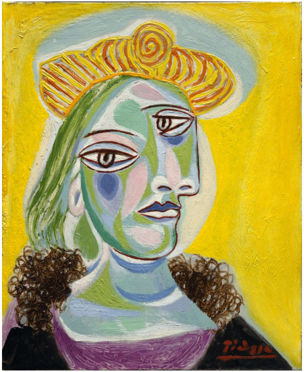
- distances
- corners or edges
- size
- any sort of differential structure (tangents, normals...)

Topology is good to describe:

- The general shape of an object.
- The number of components or holes.

Picasso is a Topologist?





The Homeomorphy question

Given two topological spaces X, Y, we can ask wether they are homeomorph.

If yes $\Rightarrow$ give an explicit homeomorphism!

If no $\Rightarrow$? We need obstructions to the existence of such a homeomorphism!

Topology should be reasonably powerfull to distinguish objects of different dimensionality!

Conjecture

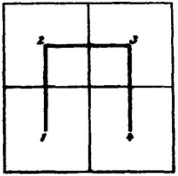
$$\mathbb{R}^n \ncong \mathbb{R}^m$$

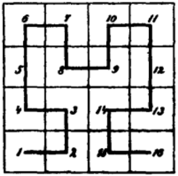
Spacefilling curves

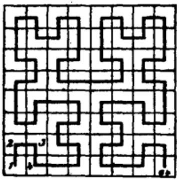
Definition

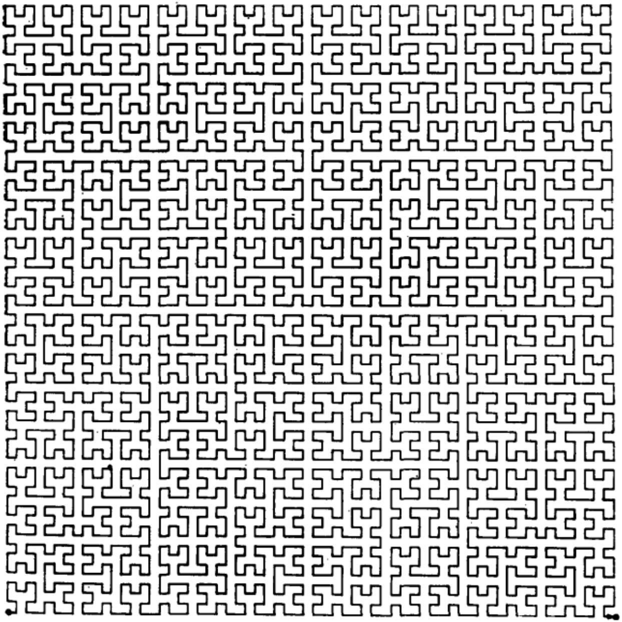
Let $I = [0, 1] \subset \mathbb{R}$ be the unit interval. We call a continuous, surjective map $\gamma : I \to I \times I$ a **spacefilling curve**.

Does such a map exist? Peano, Hilbert and others began experimenting with such curves in the end of 19.century.









A short reminder of *n*-ary numbers

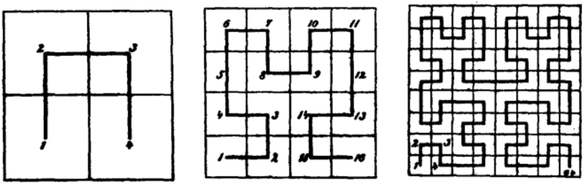
The way we represent any number x is by choosing a base b and digits $d_k \in 0, 1, ..., b-1$ such that

$$x = \sum_{k=0}^{N} d_k b^k$$

Particularly we use bases $b = 10$ (**decimal**), $b = 2$ (**binary**), $b = 16$ (**hexadecimal**).
In our case we need base $b = 4$, hence digits 0,1,2 and 3.

$$\begin{array}{rcl} 1 \equiv 1 & 5 \equiv 11 & 1/4 \equiv 0.1 \\ 2 \equiv 2 & 10 \equiv 22 & 1/16 \equiv 0.01 \\ 3 \equiv 3 & 50 \equiv 3 * 16 + 2 = 302 & 1/3 \equiv 0.\bar{1} \\ 4 \equiv 10 & 100 \equiv 64 + 2 * 16 + 4 = 1210 & \pi \equiv 3.0210033312222.... \end{array}$$

The Hilbert curve



Theorem

There exists a space filling curve.

The Hilbert curve

Proof.

We disect the square $I \times I$ in 4 parts I_0, I_1, I_2, I_3. We continue this process as indicated in the picture above to get 4^n little squares $I_{d_1 d_2 \dots d_n}$ with $d_k \in \{0, 1, 2, 3\}$. Hence for every point in $I \times I$ we get an infinite sequence of digits $d_1 d_2 \dots . d_n \dots .$. On the other hand every number x in $I = [0, 1]$ has a representation in base 4

$$x = \sum_{k=1}^{\infty} d_k 4^{-k}$$

Define $\gamma : I \to I \times I$ by $\gamma(x) = \bigcap_{k=1}^{\infty} I_{d_1 \dots d_k}$ the unique point with the same digit expansion. This can be shown to be well defined. (Digit expansions are not unique!) □

The Hilbert curve

Proof.

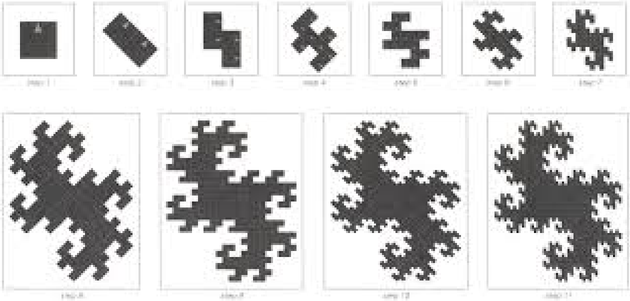
γ is continuous:

$$\forall \epsilon > 0 \exists \delta > 0 : \forall t_1, t_2 |t_1 - t_2| \leq \delta \Rightarrow |\gamma(t_1) - \gamma(t_2)| \leq \epsilon$$

Given an ϵ choose n such that $\sqrt{5}/2^n \leq \epsilon$ and let $\delta = 1/4^n$. If $|t_1 - t_2| \leq 1/4^n$ that mean that there digit expansion only varies after the $(n-1)$-th digit and the n-th digit varies at most by 1. Hence there images are in consecutive squares of sidelength $1/2^n$. The diagonal of this rectangle is $\sqrt{5}/2^n$ hence $|\gamma(t_1) - \gamma(1_2)| \leq \epsilon$. □

n-ary numbers and fractal geometry

The Hilbert curve is a so called **iterated function system**, which can be used to generate fractals. There exist a deep connection between these shapes and number systems. For example: complex numbers in base $-i-1$ with digits 0,1 and only negative exponents look like this:



"Measure, Topology, and Fractal Geometry", Gerald Edgar [1]
"Intersections of the Twin Dragon with rational lines", Paul Großkopf, TU Wien [2]

The end of topology?

Is there a homeomorphism $\mathbb{R} \to \mathbb{R}^m$?

Theorem

$$\mathbb{R} \ncong \mathbb{R}^m$$

Proof.

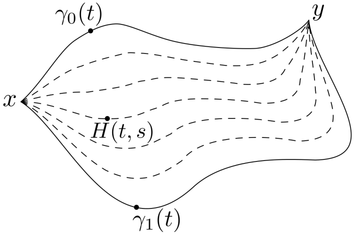
Suppose $\mathbb{R} \cong \mathbb{R}^m$ via a homeomorphism ϕ. Then $\mathbb{R} \setminus 0 \cong \mathbb{R}^m \setminus \phi(0)$. But $\mathbb{R} \setminus 0$ is not connected, $\mathbb{R}^m \setminus \phi(0)$ still is connected. A homeomorphism should preserve these properties. Hence we get a contradiction! $\square$

This argument fails to proof $\mathbb{R}^n \ncong \mathbb{R}^m$ for $n \geq 2$!

More powerful tool

Definition

Let $f, g : X \to Y$ two continous maps. We call f and g **homotopic**, if there is a continuous map $H : X \times I \to Y$ such that $H(x, 0) = f(x), H(x, 1) = g(x)$. We call H a **homotopy** and write $f \simeq g$.



Proposition

Homotopy is an equivalence relation.

Example

Example

Let X be the 1-sphere $\mathbb{S}^1 = \{x \in \mathbb{R}^2 : ||x|| = 1\}$ and Y the 2-disc $\mathbb{D}^2 = \{x \in \mathbb{R}^2 : ||x|| \leq 1\}$. Let $f(x) = x$ be the inclusion $\mathbb{S}^1 \hookrightarrow \mathbb{D}^2$ and $g(x) = 0$ the constant map sending everything to the origin. Then $H : \mathbb{S}^1 \times I \to \mathbb{D}^2$ with $H(x, t) = tx$ is a homotopy between f and g.

Homotopy equivalences

Definition

Given two topological space X, Y. We call them **homotopy equivalent** , if there exists a maps $f : X \to Y$ and $g : Y \to X$ with $fg \simeq \text{Id}_Y$ and $gf \simeq \text{Id}_X$.

Example

Let $X = \mathbb{R}^n$ and $Y = 0$ the origin. Let $f : X \to Y$ be the constant zero map and $g : Y \hookrightarrow X$ the inclusion. Then $H(x, t) = tx$ is a homotopy from Id_X to gf and $fg = \text{Id}_Y$. We call X **contractible** if its homotopy equivalent to a point.

Homotopy equivalences

Example

The annulus $X = \{x \in \mathbb{R}^2 : 1 \leq ||x|| \leq 2\}$ is homotopy equivalent to the circle with radius 1.5 $Y = \{x \in \mathbb{R}^2 : ||x|| = 1.5\}$.

Example

Let $X = \mathbb{R}^{n+1} \setminus 0$ and $Y = \mathbb{S}^n$ the unit sphere. Then $f : X \to Y$ with $f(x) = x/||x||$ and the inclusion $g : Y \hookrightarrow X$ form a homotopy equivalence via

$$H(x, t) = tx + (1 - t)\frac{x}{||x||}$$

A topologists alphabet

Alphabet:
A B C D E F G H I J K L M N O P Q R S T U V W X Y Z
a b c d e f g h i j k l m n o p q r s t u v w x y z
Alphabet (Topologist Version (with meme)):
O B I O I I I I I I I I I O O B O I I I I I I I I I
o o l o o l g l i i l l l l 6 9 o l l l l l l l l l

Equivalence classes of the English (i.e., Latin) alphabet (sans-serif)

Homeomorphism	Homotopy equivalence
{A, R} {B} {C, G, I, J, L, M, N, S, U, V, W, Z} {D, O} {E, F, T, Y} {H, K} {P, Q} {X}	{A, R, D, O, P, Q} {B} {C, E, F, G, H, I, J, K, L, M, N, S, T, U, V, W, X, Y, Z}

The fundamental group

Definition

Let $f, g : I \to X$ two continuous paths in X. We call f and g **homotopic relative to endpoints**, if there is a continuous map $H : I \times I \to X$ such that $H(x, 0) = f(x), H(x, 1) = g(x)$ and
$H(0, t) = g(0) = f(0), H(1, t) = g(1) = f(1)$. We write $f \simeq_{\partial I} g$ or $f \simeq g$.

Definition

Let X be a topological space and $x_0 \in X$. Let
$\Omega(X, x_0) := \{f : I \to X | f(0) = f(1) = x_0\}$ the set of continuous **loops** in X. Homotopy relative to endpoints induces an equivalence realtion on $\Omega(X, x_0)$ and we can define the **fundamental group**

$$\pi_1(X, x_0) := \Omega(X, x_0)/_{\simeq}$$

The group operation in $\pi_1(X, x_0)$

Theorem

The fundamental group is a group!

We define the multiplication as the concatenation of loops. Let f, g be two loops in X, then

$$g \circ f := \begin{cases} f(2t) & t \in [0, 1/2] \\ g(2t-1) & t \in [1/2, 1] \end{cases} \tag{1}$$

Now $[g] \circ [f] := [g \circ f]$ for equivalence classes in $\pi_1(X, x_0)$. The constant loop $[x_0]$ is the unit element of this operation. The inverse is given by

$$\bar{f}(t) := f(1-t),$$

the path that goes in the inverse direction and $[f]^{-1} = [\bar{f}]$. This structure can be extended to paths, that can be concatenated!

The First Kiss

We can look at the fundamental group as an asignement from the collection of pointed Topological spaces Top_* to the collection of groups Grp

$$(X, x_0) \mapsto \pi_1(X, x_0)$$

Further for any continuous map $\phi : X \to Y$ we get a group homomorphism $\pi_1(\phi) : \pi_1(X, x_0) \to \pi_1(Y, \phi(x_0))$ by $\pi_1(\phi)[f] = [\phi f]$
Categorically speaking we have a **functor** between the categories Top_* and Grp.

Invariance of the base point

Theorem

Let X be a pathconnected topological space and $x_0, y_0 \in X$. Then $\pi_1(X, x_0) \cong \pi_1(X, y_0)$

Proof.

We can choose a path g form x_0 to y_0. The isomorphism $\pi_1(X, x_0) \to \pi_1(X, y_0)$ is given by conjugation with g, meaning for any loop f from x_0 to itself we map it to the loop gfg^{-1} from y_0 to itself. The inverse $\pi_1(X, y_0) \to \pi_1(X, x_0)$is given by conjugation by g^{-1} □

We therefore can omit the base point for path connected X and write $\pi_1(X)$

Homotopy invariance

Theorem

Let $\phi : X \to X$ be homotopic to the identity $\phi \simeq \mathrm{Id}_X$. Then $\pi_1(\phi) : \pi_1(X, x_0) \to \pi_1(X, \phi(x_0))$ is an iso.

Proof.

Let $H : X \times I \to Y$ the homotopy between Id_X and ϕ. Then $H(x_0, .) : I \to X$ is a path g in X from x_0 to $\phi(x_0)$. $\pi_1(\phi)$ is given by conjugation with this path.

$$\pi_1(\phi)[f] = [\phi f] = [gfg^{-1}]$$

This is again an iso. □

Examples

Corrolary

Given a homotopy equivalence of path connected spaces $X \simeq Y$, $f : X \to Y$ *and* $g : Y \to X$ *with* $fg \simeq \mathrm{Id}_Y$ *and* $gf \simeq \mathrm{Id}_X$, *we get that*

$$\pi_1(X) \cong \pi_1(Y)$$

Example

The singleton space has trivial fundamental group $\pi_1(\{*\}) = 1$, hence any contractible space has trivial fundamental group too. Particularly $\pi_1(\mathbb{R}^n) = 1$.

Example

The annulus is homotopy equivalent to the circle hence they have the same fundamental group. Further: $\pi_1(\mathbb{R}^{n+1} \setminus 0) = \pi_1(\mathbb{S}^n)$.

The fundamental group of the circle

Theorem

$$\pi_1(\mathbb{S}^1) = \mathbb{Z}$$

Proof.

We show this in two step:

1. Any loop is homotopic to $z^n : t \mapsto e^{2n\pi it}$, for some $n \in \mathbb{Z}$. (Surjective)
2. The z^n, z^m are not homotopic for $n \neq m$. (Injective)

□

The fundamental group of the circle

Proof.

Let $U_1 = \{(x_1, x_2) \in \mathbb{S}^1 : x_2 \geq 0\}$ and $U_2 = \{(x_1, x_2) \in \mathbb{S}^1 : x_2 \leq 0\}$, hence $U_1 \cap U_2 = \{(1,0), (-1,0)\}$. Let $f : I \to \mathbb{S}^1$ be a loop based at $(1,0)$ then I can be disected into n subintervals $I_k = [t_k, t_{k+1}]$ such that:

- $f|_{I_k}$ has image either in U_1 or U_2
- The images of two consecutive intervals only intersect in $\{(1,0), (-1,0)\}$

Hence f can be written as the product of paths $f_n \cdots f_2 f_1$. □

The fundamental group of the circle

Proof.

These paths f_n are maps $I \to U_i \cong I$ so there are only 6 possible homotopy types:

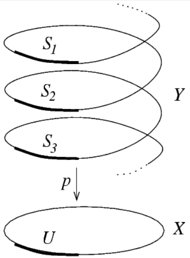
- Constant maps to $(1,0)$ or $(-1,0)$ denoted by 1
- A simple path in U_1 from $(1,0)$ to $(-1,0)$ or its inverse denoted by δ_1 ad δ_1^{-1}.
- A simple path in U_2 from $(-1,0)$ to $(1,0)$ or its inverse denoted by δ_2 ad δ_2^{-1}.

Notice that $[\delta_2\delta_1] = [z^1]$. Since paths can only concatenated if they meet in the same point $[f]$ can be simplified to either 1, $[\delta_2\delta_1 \cdots \delta_2\delta_1]$ or $[\delta_1^{-1}\delta_2^{-1} \cdots \delta_1^{-1}\delta_2^{-1}]$ hence there exists a number $n \in \mathbb{Z}$ such that $[f] \simeq [z^n]$. □

The fundamental group of the circle

Proof.

To show that these simple loops are not homotopic in $\mathbb{S}^1$ we use **covering spaces** and **liftings**. Without proof we use that the map $\exp : t \mapsto e^{2\pi it}$ maps $\mathbb{R}$ onto $\mathbb{S}^1$ and for any loop f based at $(1, 0)$ we get a unique map $\tilde{f} : I \to \mathbb{R}$ such that $\tilde{f}(0) = 0$ and $\exp \circ \tilde{f} = f$. Particularly $\tilde{f}(1) \in \mathbb{Z}$. Further homotopic loops are maped to homotopic paths up to Endpoints. It is easy to compute that z^n is lifted to multiplication by n hence $\tilde{z^n}(1) = n$. Therefore $z^m \simeq z^n$ implies $\tilde{z^m} \simeq \tilde{z^n}$ and particularly $\tilde{z^m}(1) = m = n = \tilde{z^n}(1)$. □



The fundamental group of higher spheres

Theorem

$$\pi_1(\mathbb{S}^n) = 1, n \geq 2$$

Proof.

Let f be a loop in $\mathbb{S}^n$ and $x \in \mathbb{S}^n$ not in the image of f. Then $\mathbb{S}^n \setminus \{x\} \cong \mathbb{R}^n$ by stereographical projection. We know that $\pi_1(\mathbb{R}^n) = 1$, hence f is homotopic to the constant loop in $\mathbb{S}^n \setminus \{x\}$. Therefore it is also homotopic to the constant loop in $\mathbb{S}^n$. □

Notice that we cheated here by assuming $f(I) \neq \mathbb{S}^n$! Since we already saw space filling curves this can be the case, but we always can homotopically deform any curve away from a small circle.

A step closer

Theorem

$$\mathbb{R}^2 \ncong \mathbb{R}^m, m \geq 3$$

Proof.

Suppose $\mathbb{R}^2 \cong \mathbb{R}^m$ via a homeomorphism ϕ. Then $\mathbb{R}^2 \setminus 0 \cong \mathbb{R}^m \setminus \phi(0)$. But $\mathbb{R}^2 \setminus 0 \equiv \mathbb{S}^1$ has nontrivial fundamental group, $\mathbb{R}^m \setminus \phi(0) \equiv \mathbb{S}^{m-1}$ has trivial fundamental group. A homeomorphism should preserve these properties. Hence we get a contradiction! □

This argument fails to proof $\mathbb{R}^n \ncong \mathbb{R}^m$ for $n \geq 3$! $\Rightarrow$ Use Higher Homotopy groups!

The Marriage

The techniques developed here can be extended in the **field of algebraic topology**. It explores **higher homotopy groups** which can be seen as homtopy types of maps $\mathbb{S}^n \to X$ generalizing loops.
A more abstract, but more computable approach is **homology**. Here we also have a functor form Top to an algebraic category like groups Grp or algebras Alg .
Further beautiful things like **cohomology, stable homology, spectral sequences, K-Theory....**
"Homotopical Topology", Anatoli Formenko, Dimitri Fuchs, Springer Verlag.
"Algebraic Topology", Allen Hatcher:
https://pi.math.cornell.edu/ hatcher/AT/AT.pdf

The higher homotopy groups of spheres

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbf{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{12}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_3$	$\mathbf{Z}_{15}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_{12}\times\mathbf{Z}_2$	$\mathbf{Z}_{84}\times\mathbf{Z}_2^2$	$\mathbf{Z}_2^2$
S^3	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{12}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_3$	$\mathbf{Z}_{15}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_{12}\times\mathbf{Z}_2$	$\mathbf{Z}_{84}\times\mathbf{Z}_2^2$	$\mathbf{Z}_2^2$
S^4	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}\times\mathbf{Z}_{12}$	$\mathbf{Z}_2^2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_{24}\times\mathbf{Z}_3$	$\mathbf{Z}_{15}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^3$	$\mathbf{Z}_{120}\times\mathbf{Z}_{12}\times\mathbf{Z}_2$	$\mathbf{Z}_{84}\times\mathbf{Z}_2^5$
S^5	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{30}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^3$	$\mathbf{Z}_{72}\times\mathbf{Z}_2$
S^6	0	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_{60}$	$\mathbf{Z}_{24}\times\mathbf{Z}_2$	$\mathbf{Z}_2^3$
S^7	0	0	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	0	0	$\mathbf{Z}_2$	$\mathbf{Z}_{120}$	$\mathbf{Z}_2^3$
S^8	0	0	0	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	0	0	$\mathbf{Z}_2$	$\mathbf{Z}\times\mathbf{Z}_{120}$

The homotopical proof of the fundamental theorem of Algebra

Theorem (The Fundamental Theorem of Algebra)

Any polynomial $P \in \mathbb{C}[x]$ with coefficients in the field of complex numbers $\mathbb{C}$ of degree $n \geq 1$ has a root in $\mathbb{C}$.

We proof this indirectly. So assuming the image of P is a subset of $\mathbb{C} \setminus 0$ we consider restrictions of P to concentric circles with radius R.
$P|_{C_R} : C_R \to \mathbb{C} \setminus 0$ can be considered a loop in $\mathbb{C} \setminus 0 \cong \mathbb{R}^2 \setminus (1,0) \simeq \mathbb{S}^1$.
We therefore want use the properties of the fundamental group $\pi_1(\mathbb{S}) \cong \mathbb{Z}$.

Animation in Matematica

The homotopical proof of the fundamental theorem of Algebra

Proof.

Let wlog. $P(z) = z^n + ... + a_1 z + a_0$ and suppose that $\operatorname{Im}(P) \subset \mathbb{C} \setminus 0$. Let $C_R := \{x \in \mathbb{C} : |x| = R\}$ and $P_R(t) := P(R \cdot z(t))$ the loop defined by the image of P_{C_R}, the restriction of the polynomial to this circle. Since polynomial eventually behave like their leading term we want to proof that there is an R sufficiently large such that $P_R \simeq z^n$. Let $R \in \mathbb{R}$ such that for all $|z| = R$ we have $|z|^{n-j} > 2n|a_j|$. Equivalentely

$$\frac{|z^n|}{2n} > |a_j z^j|$$

□

The homotopical proof of the fundamental theorem of Algebra

Proof.

Hence

$$\frac{|z^n|}{2} > |a_{n-1}z^{n-1}| + ... + |a_1 z| + |a_0| \geq |a_{n-1}z^{n-1} + ... + a_1 z + a_0|$$

Then for $|z| = R$

$$\begin{aligned} |P(z)| &= |z^n + ... + a_1 z + a_0| \\ &\geq |z^n| - |a_{n-1}z^{n-1} + ... + a_1 z + a_0| \\ &= \frac{|z^n|}{2} + \left(\frac{|z^n|}{2} - |a_{n-1}z^{n-1} + ... + a_1 z + a_0|\right) \\ &> \frac{|z^n|}{2} \end{aligned}$$

□

The homotopical proof of the fundamental theorem of Algebra

Proof.

We define the homotopy $H : I \times I \to \mathbb{C}$ by $H(x,t) = tR^n z^n(x) + (1-t)P_R(x)$ where $z^n(x) := e^{2\pi i n x}$. Now

$$\begin{aligned} |H(x,t)| &= |(R^n z^n + ... + a_1 Rz + a_0) \\ &- t(a_{n-1}R^{n-1}z^{n-1} + ... + a_1 Rz + a_0)| \\ &\geq \frac{|R^n|}{2} - |a_{n-1}R^{n-1}z^{n-1} + ... + a_1 Rz + a_0| \\ &> 0 \end{aligned}$$

So this is a homotopy from $R^n z^n$ to P_R in $\mathbb{C} \setminus 0 \simeq \mathbb{S}^1$! Hence $[P_R] = n \in \pi_1(\mathbb{S}) \cong \mathbb{Z}$. □

The homotopical proof of the fundamental theorem of Algebra

Proof.

On the other hand P_R is homotopic to the constant map a_0 by $H(x, t) = P(tR \cdot z(x))$ because P is continuous and we assumed that it has no roots. Therefore $[P_R] = 0 \in \pi_1(\mathbb{S})$ hence we get a contradiction. □

Thank you for your attention!
I am looking forward to your questions!

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